# Studying the Relationships between Spatial Structures of Wireless Networks and Population Densities

Maria Michalopoulou, Janne Riihijärvi and Petri Mähönen
RWTH Aachen University, Institute for Networked Systems
Kackertstrasse 9, D-52072 Aachen, Germany
Email: {mmi, jar, pma}@inets.rwth-aachen.de

***Abstract*—In this paper we show how to quantify dependency between the node distributions of wireless networks and the underlying population densities. Furthermore, we show that a quantitative analysis of this relation can be beneficial for understanding and generating realistic network models. Towards this direction we argue that spatial statistics is an appropriate method for this purpose. As a case study we analyze the correlations of GSM-900 and UMTS base stations, with the underlying population densities of Germany. We find that there is a significant statistical similarity between the locations of GSM-900 base stations and population, due to the immense penetration of the network, and a less tight relation in the case of UMTS network. We consider the problem of covering a given population pattern and use the concept of fractal dimension to show how the number of required base stations changes when their maximum coverage range decreases, according to the population pattern.**

## I. INTRODUCTION

Over the past years it has been recognized that the assumption of randomly distributed nodes, sampled from a uniform distribution, is rather implausible. However, due to the lack of understanding and quantitative discussion regarding the node distributions in deployed wireless networks, the majority of the existing research and analysis is based on this assumption. Moreover, we know that the spatial distribution of the nodes is an important issue; several studies, for example, [1], [2], [3], have shown that the node distribution has a significant impact on several properties of the network, such as capacity, throughput, and connectivity. This fact indicates that there is a clear need for revising the network topologies used in research studies and network analyses.

Especially for commercial wireless networks deployed in large geographical areas, the assumption of uniformly distributed nodes is definitely unrealistic. Cellular operators have known this for a long time, but in the research community quantitative analysis on the node correlations has been almost non-existent. It is reasonable to assume that node locations probably depend on underlying phenomena like population densities. The aim of commercial networks is to provide services to users, who correspond naturally to a part of the population. Therefore, the existence of an interaction between the spatial structure of wireless networks and population densities is a logical hypothesis. We would further expect that the degree of this interaction is dependent upon several factors, such as the income levels of the population, the type of the network, the degree of penetration of the offered services, and the stage of deployment of the network. For example, network types that provide services for distinct users, such as the 2G and 3G cellular networks, need to support individual transmissions for different users in both directions, uplink and downlink. In such cases the network dimensioning is performed in terms of capacity requirements since it needs to take into account the amount of offered traffic. Thus, the number of required base stations within a specific area is expected to increase with the number of users since more users are likely to generate more traffic load. On the contrary, a broadcasting network is mainly concerned with the geographical coverage of the populated areas, and therefore our initial hypothesis is not as strong as in the first example. Another relevant factor is the degree of penetration of the offered services, or alternatively, the stage of deployment of the network. A very high level of coverage, e.g., as in the case of 2G cellular networks, indicates that the distribution of users in a large area shall follow closely the measured population densities. Regarding the stage of deployment, a network that is still not fully deployed, for instance, a 3G cellular network, shall exhibit a less tight relation with the underlying population distribution. But obviously one would expect that, regardless of the network type, underlying spatial phenomena like population densities and income levels do not describe the correlations of node distributions for very short distances since other local dependencies such as propagation characteristics start to dominate.

In order to characterize quantitatively the interaction between wireless networks and population densities we propose the employment of *spatial statistics*. Although a relationship clearly exists between them, to the best of our knowledge it has not yet been studied *quantitatively* in the existing literature. The major advantage of spatial statistics is that they can be used to study spatial correlation between different phenomena. This study builds upon our earlier work on spatial statistics of wireless networks (see, for example, [4], [5]). As a case study, we have performed a comparative spatial correlation analysis between the locations of GSM-900 and UMTS base stations, deployed in Germany by one of the major operators, and the underlying population densities.

We consider also an application emerging from the influence of population distribution on network structure. Specifically, we show how the number of base stations required to cover a given population pattern is affected by the spatial structure of the given pattern. We use the *fractal dimension* of the population pattern for estimating the number of required base stations when decreasing the maximum coverage range. This application could provide useful practical estimations; for instance, it may be exploited for estimating the number of needed micro or picocells from the number of macrocells.

The rest of this paper is organized as follows. In Section II we give a brief overview of spatial statistics and introduce the methods we have used in our study. The case studies and the corresponding results are presented in Section III. In Section IV we describe our application indicating the quantitative influence of the underlying population pattern on the network structure. Finally, the paper is concluded in Section V.

## II. Spatial Statistics

For investigating the statistical similarity between the spatial distribution of wireless networks and population densities we employ tools of spatial statistics. In order to provide the necessary background, we will present an overview of spatial statistics within the context of addressing wireless networks as *spatial point processes*. We shall also introduce the methods and estimators we have used. For a more comprehensive discussion on the subject the reader is referred to [6], [7].

Spatial statistics is a branch of applied statistics dealing with the analysis of spatial data, such as locations of wireless network nodes or population distributions. A spatial random process in a region of study $A$ is a collection of random variables $\{X(x)|x \in A\}$, where $X$ indexes locations in $A$. The set $X(x)$ can be viewed as a randomly realized surface over the region and is termed a spatial random field. In practice, this surface is only observed at a finite set of locations. In this context a set of points can be interpreted as a spatial point process. Given the locations of base stations defined in a two dimensional area, we consider that the wireless network is represented by a two-dimensional spatial point process.

The fundamental characteristic of a spatial point process is its intensity function, $\lambda(x)$, which is proportional to the point density around a location $x$. If $dx$ denotes an infinitesimal area around a location $x$, then $\lambda(x)dx$ is the probability that there is one point in $dx$. Under the assumption of stationarity, the density function is equal to a constant quantity $\lambda$, which is termed intensity and is equal to the mean number of points per unit area.

Expanding the concept of the intensity function $\lambda(x)$, we introduce the second-order intensity function, $\rho^{(2)}(x_1, x_2)$, which considers spatial interactions between pairs of points. Let us consider two disjoint infinitesimal areas $dx_1$ and $dx_2$ around the locations $x_1$ and $x_2$ respectively. Then, $\rho^{(2)}(x_1, x_2)dx_1\ dx_2$ is the probability that there is one point in each of the two areas. In case of a stationary and isotropic process a function that depends upon $x_1$ and $x_2$ can be reduced to a function of the distance $r$ between $x_1$ and $x_2$. Therefore, the second-order intensity function of a stationary and isotropic process can be denoted by $\rho^{(2)}(r)$. Without losing generality we shall be considering in this paper only stationary and isotropic spatial point processes defined in a two-dimensional space.

In this paper we present a spatial analysis by means of the *pair correlation* and *cross correlation* functions. The correlations between node locations are expressive and practical tools for characterizing point distributions.

### A. The Pair Correlation and Cross Correlation Functions

The pair correlation function $\xi(r)$ measures the excess probability of finding a neighboring node at distance $r$ from a given node. Assuming two circles of infinitesimal areas $dA_1$ and $dA_2$ whose centers are separated by distance $r$, the probability of finding one point in each of the circles is given by

$$p_r = \lambda^2(1 + \xi(r))dA_1\ dA_2, \tag{1}$$

which is also equivalent to

$$\rho^{(2)}(r) = \lambda^2(1 + \xi(r)). \tag{2}$$

The above equation illustrates that the pair correlation function can be fully derived from the first- and second-order intensity functions. For this reason it is referred to as a second-order method of spatial statistics.

For a Poisson point process, i.e., a uniformly random process, the pair correlation function is equal to zero for all values of $r$. Values of $\xi(r)$ larger than zero indicate that for the corresponding $r$ the interpoint distances are more frequent than they would be in a completely random process; this is an evidence of clustering if it occurs for relatively small values of $r$.

Similar definition can be given for cross correlation function, $\xi_{ab}(r)$. Without loss of generality we define a situation where each point belongs either to a class $a$ or class $b$. Then the cross correlation function measures the symmetric probability of finding a node of class $a$ at distance $r$ from a node of class $b$, that is

$$p_{r_{ab}} = \lambda_a\lambda_b(1 + \xi_{ab}(r))dA_1\ dA_2, \tag{3}$$

where $\lambda_a$ and $\lambda_b$ are the densities of the two classes of points. The notion of points belonging to different classes can be easily generalized to interpret several different scenarios; it can also represent points than belong to two different point processes.

For estimating the correlation functions we shall follow the long line of solutions developed by the astrophysics community. These estimators are based on a pair counting method; for every point $x$ in the point process we count the number of neighbors lying in an annulus with infinitesimal width $dr$, centered at $x$. First, by using the following notation we define the pair counting function [8]

$$\Phi_r(x, y) = \big[r - dr/2\ \leq\ d(x, y)\ \leq\ r + dr/2\big], \tag{4}$$

where $d(x, y)$ denotes the distance between two points $x$ and $y$ and the Iverson bracket $[P]$ equals one if $P$ is true, and 0 otherwise. Secondly, given the point process $D$ of $N$ points we generate a random point distribution $R$ of $M \gg N$ random points by drawing them from a distribution that does not exhibit any correlations. Then we form the normalized pair counts

$$DD(r) = \sum_{x \in D} \sum_{y \in D} [x \neq y] \Phi_r(x, y)/(N(N-1)), \quad (5)$$

$$DR(r) = \sum_{x \in D} \sum_{y \in R} \Phi_r(x, y)/(NM), \quad (6)$$

$$RR(r) = \sum_{x \in R} \sum_{y \in R} [x \neq y] \Phi_r(x, y)/(M(M-1)). \quad (7)$$

The purpose of introducing the random point process $R$ is to eliminate the edge effects. Finally, we can write the Landy and Szalay (LS) estimator [9] for the pair correlation function of the given point process $D$

$$\xi_{LS} = (DD - 2DR + RR)/RR. \quad (8)$$

Following the same approach, the most widely used estimator for cross correlation function is the one by Peebles [10]. We denote by $D_a$ and $D_b$ the data points in $D$ belonging to classes $a$ and $b$ respectively, and we obtain the following normalized pair counts

$$D_a D_b(r) = \sum_{x \in D_a} \sum_{y \in D_b} \Phi_r(x, y)/N_a N_b, \quad (9)$$

$$D_a R(r) = \sum_{x \in D_a} \sum_{y \in R} \Phi_r(x, y)/N_a M. \quad (10)$$

Then, we may write the Peebles estimator

$$\xi_{ab,Peeb} = D_a D_b / D_a R - 1. \quad (11)$$

## III. Case Study for Cellular Base Stations

In this section we present the results of the spatial statistical analysis we performed for cellular base stations in Germany. For this study a sample of 19624 GSM-900 and 1834 UMTS base stations was chosen. We also analyzed the underlying population densities measured on a grid resolution of approximately 900 m$\times$900 m, as they are obtained from the Landscan dataset (2006 release) [11].

We generated spatial point processes that follow the measured population densities and have a number of points approximately equal to the number of base stations. In order to achieve this we first diminished the measured population densities in each grid cell by a constant factor, in a way that the expected value of the overall population is equal to the desired number of points, and then according to the diminished values we generated a Poisson point process within each cell. The outcome can also be interpreted as an inhomogeneous Poisson process with intensity function, $\lambda(x)$, proportional to the population densities. Figure 1 shows two different realizations with approximately 20000 and 50000 points respectively. In all cases the results involving population patterns are calculated by 10 realizations of similarly generated point patterns. The resulting values are the mean values of the 10 estimations; error bars are omitted in the cases where the standard deviation was extremely small.

(a) (b)

Fig. 1: Point patterns resulting from the measured population densities of Germany (a) 20000 points (b) 50000 points.

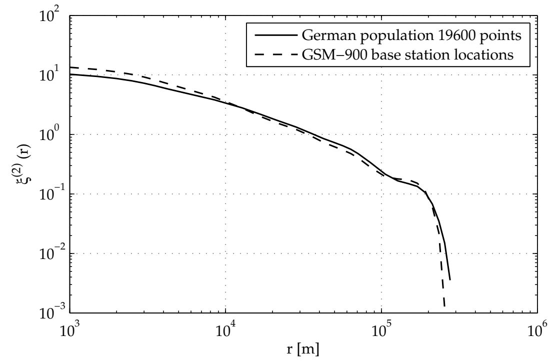



Fig. 2: The pair correlation function for the GSM-900 base stations in Germany and a point process of approximately 19600 points following the population distribution.

Figure 2 shows the pair correlation functions, calculated by the LS estimator, for the GSM-900 transceivers and for population patterns of approximately 19600 points. We note for reference that the pair correlation function of a point process that does not exhibit correlations is 0 for all values of $r$. Thus, the pair correlation functions of Figure 2 illustrate that the point distributions of the two analyzed point processes are far from being uniformly distributed. Moreover, there is a considerable statistical similarity between them. The fact that the spatial distributions of the two point processes are not independent is also verified by the cross correlation function between the two point processes, which is presented in Figure 3. The cross correlation function between two uncorrelated point distributions would be equal to 0 for all values of $r$. For very long distances it is natural that the cross correlation function tends to 0 since processes usually tend to become uncorrelated.

It is especially remarkable that the pair correlation functions

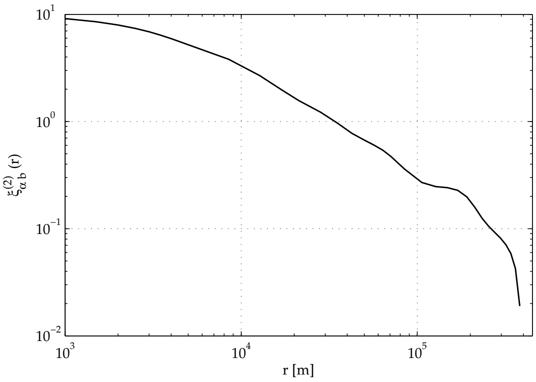



Fig. 3: The cross correlation function between the GSM-900 base stations in Germany and a point process of approximately 19600 points following the population distribution.

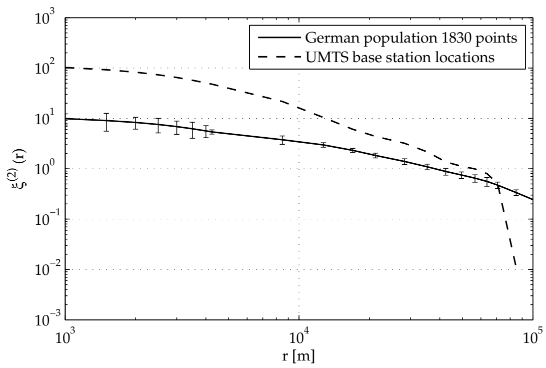



Fig. 4: The pair correlation function for the UMTS base stations in Germany and a point process of approximately 1830 points following the population distribution.

of the two point processes (Figure 2) are almost identical. We should not underestimate the fact that, as discussed in Section I, this is mainly due to some characteristics of the network under consideration, namely of the GSM-900 network. First, the network needs to meet not only coverage, but also capacity requirements. As a second, the GSM services are highly used by the majority of the population, hence the amount of offered traffic in a given location is expected to be strongly related to the population density. In order to stress the effect of these features, it is interesting to observe the pair correlation functions for UMTS network. Presently, the UMTS network is not very broadly utilized and, unlike the GSM, does not provide coverage to the entire area of Germany. The resulting pair correlation functions are shown in Figure 4. We observe that there is a considerable gap between the pair correlation of the population patterns and the pair correlation of the UMTS base stations. We note that the standard deviation for different realizations of population patterns is not negligible here due to the fact that the number of points in the population patterns, which are chosen to be approximately equal to the number of base stations, is smaller in this case.

## IV. Estimating Changes in Network Structure Using Fractal Dimension of Population Patterns

In order to illustrate an example of how the population distribution can influence the network structure, we present a simple case study. We use the concept of fractal dimension for describing a population pattern, and we estimate the number of required base stations needed to cover the given pattern when changing their maximum coverage radii. We argue that this application can exploit its usefulness in many practical cases; for example, in network planning, in applying cell splitting, or in theoretical estimations. Before presenting the results we shall give a brief theoretical overview of the involved concepts and methods.

The fractal dimension is one of the main concepts of fractal geometry [12]. Conventionally, we would say that the dimension of a set of disconnected points is 0, the dimension of a smooth curve is 1, the dimension of a surface is 2, and so on. These are the so called topological dimensions of objects. But it is possible to consider other forms of dimension with different definitions and allow values that are not necessarily integers. Loosely speaking, the aim of fractal dimension is to provide a description of how much space a set fills. It is a measure of the prominence of the irregularities of a set when viewed at small scales. Therefore, halving the maximum base station range does not necessarily result in four times more base stations needed, but the new number of base stations depends precisely on the fractal dimension.

There are many alternative mathematical definitions of fractal dimension. Here we employ the *box-counting dimension*, which is widely used in practice due to its relative ease of estimation. Let $B$ be any non-empty bounded subset of $R^n$, and let $N_\delta(B)$ be the smallest number of sets of diameter at most $\delta$ which can cover $B$. The lower and upper box-counting dimensions of $B$ respectively are defined as

$$\underline{D}(B) = \varliminf_{\delta\to 0} \frac{\log N_\delta(B)}{-\log\delta}, \qquad \overline{D}(B) = \varlimsup_{\delta\to 0} \frac{\log N_\delta(B)}{-\log\delta}. \quad (12)$$

If these are equal we refer to the limit value as the box-counting dimension of $B$

$$D(B) = \lim_{\delta\to 0} \frac{\log N_\delta(B)}{-\log\delta}. \quad (13)$$

The box-counting dimension is the logarithmic rate at which $N_\delta(B)$ increases as $\delta$ tends to 0, and may be estimated by the gradient of the graph of $\log N_\delta(B)$ against $-\log\delta$.

In a practical view, let us consider the estimation of the box-counting dimension for a two-dimensional object [13]. As a first step, the object needs to be surrounded by a rectangular frame. The best way to do this is by minimizing the area of the frame. Therefore, we choose the frame to be as close as possible to the extreme points. Prior to that, depending on the shape of the object, it might be necessary to rotate it in order to achieve covering by a frame of minimum size. Actually we verified that this can have significant effect on the final result. Then the object is covered by a sequence of rectangular grids of descending cell sizes and for each of the grids, two values are recorded: the length (or width) of the rectangular cells of the grid, $s$, and the number of cells intersected by the image, $N(s)$, i.e., the number of cells that contain at least one point of the point pattern. Then, the box-counting dimension, $D$, is the slope of the following linear regression equation: $\log(N(s)) = D\log(1/s) + b$ and can be estimated from the log-log plot only if a linear relationship holds. As cells are descending in size and grids with smaller cell sizes are covering the image, the information of the picture begins to get captured and the quantity $N(s)$ approximates a limit value. Therefore, the smallest grid to work with corresponds to the last point or breakpoint for which a straight line can be approximately fitted in the first part of the plot.

### *A. Case Study*

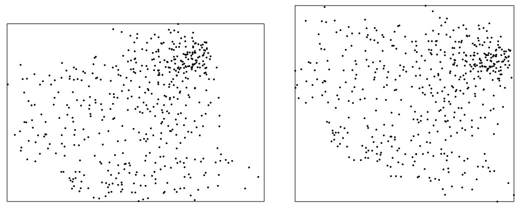
By addressing the base station placement as a plate placement problem, we attempt to estimate roughly the number of base stations required for covering a given population pattern by means of the fractal dimension of the pattern. The aim of a plate placement problem is to find the minimum number of plates (various sizes also allowed) needed to cover a given area (or point pattern), so that the number of those is minimized, the overlap is minimized, and the non covered area (or points) is minimized. Assuming a uniform radio coverage and a given maximum allowed radius $r$, our task here is to cover a given point pattern. Obviously this approach approximates the network planning from the point of view of a geometrical coverage problem (similar approaches can be found in [14], [15]).

By its definition is obvious that the box-counting dimension provides a description of how a structure can be covered by objects of the same shape. For the problem of base stations it is not difficult to find the maximum radius, $r_{max_o}$, required for covering the users with a very small number of base stations, $N_{BS_o}$, like for example 4 or 5. Then, using this as reference point we can estimate how many non overlapping base stations ($N_{BS}$) with maximum range $r_{max}$ can cover the same set of locations as follows

$$\left(\frac{r_{max_o}}{r_{max}}\right)^{-D} = \frac{N_{BS_o}}{N_{BS}}, \qquad (14)$$

where $D$ is the box-counting dimension.

For applying this concept we generated a point pattern of 500 points following the population densities of the area of San Francisco, as described in Section III. The resulting point pattern is illustrated in Figure 5a. In order to estimate the box-counting dimension the pattern was rotated in a way that the surrounding frame has a minimum size (Figure 5b). The log-log graph plotting the size of the grid cells against the number of cells containing at least one point is shown in Figure 6. The parameter $s$ is equal to the size of the grid cell, normalized to one. We fit a linear regression to the linear part of the plot and the gradient yields the box-counting dimension, which is equal to $1.91$. As starting point we found experimentally that for covering all the points with four base stations the maximum coverage range of the base stations needs to be $4250$ m. Thus,



(a) (b)

Fig. 5: A population of $500$ points in the area of San Francisco. The figure (b) shows a rotation by 10 degrees of (a) for the estimation of box-counting dimension.

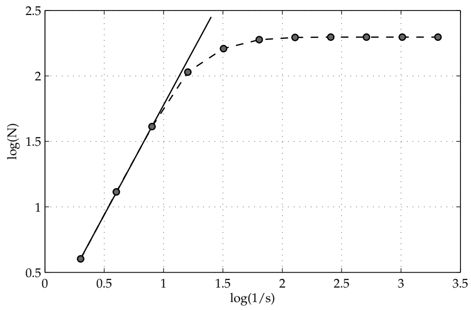



Fig. 6: Calculating the box-counting dimension; the slope of the straight line is equal to the box-counting dimension.

we have obtained a needed 'reference point'. Then, we change the maximum range of the base stations and estimate by means of equation 14 the new number of required base stations. In order to evaluate the estimated values we compare them with the values obtained by a genetic algorithm which is designed to find solution to the plate placement problem, as described above. The use of the genetic algorithm itself for this problem is not particularly interesting so we do not refer to any further details regarding its operation in this paper. The results are illustrated in Figure 7. The estimated values calculated from the box-counting dimension are indeed comparable with the values given by the genetic algorithm. In order to emphasize the actual contribution of the fractal dimension, and hence of the considered population distribution, to the estimated values we refer to [1]. In this study, Riihijärvi et. al. have also considered the base station placement as a plate placement problem and illustrated that the distribution of the points that need to be covered has a significant impact on the number of required base stations.

Moreover, we would like to stress that this case study provides an indication of the fact that results yielding from the spatial analysis of population patterns are not only of theoretical interest, but have also potential benefits in practice. For instance, this specific application could provide a preliminary estimation when needed to add new base stations to an existing network for increasing its capacity. This rough estimation can play the role of a reference value either in

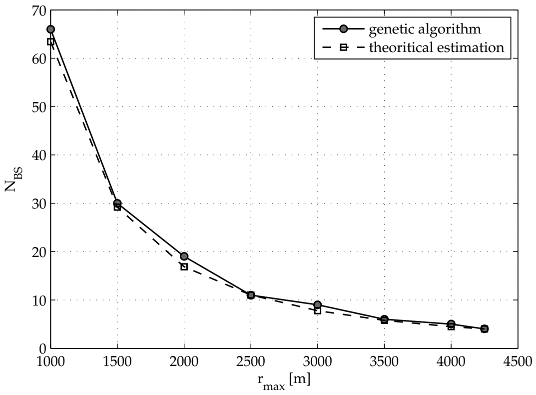



Fig. 7: The number of base stations required to cover 500 points following the population pattern of San Francisco.

theoretical evaluations performed prior to the completion of the network planning phase, or in the evaluation of base station placement solutions. It could be also used as an input to an optimization algorithm for network planning. In the majority of base station placement algorithms presented in literature the number of base stations is also a parameter to be evaluated by the algorithm. By allowing the number of base stations to vary considerably the feasible solution space grows enormously. This situation has several drawbacks. First, optimization procedures tend to consume larger amounts of time. Second, the larger the space of feasible solutions, the more difficult to design effective algorithms that can finally come up with solutions that approximate the optimal case in an acceptable amount of time. Thus, a roughly estimated value could be utilized for limiting the number of possible solutions.

## V. Conclusions

In this paper we considered the relation between node distributions in wireless networks and the underlying population densities. As a case study we have shown by means of spatial correlations that the distribution of GSM-900 base stations in Germany follows closely the population densities. Similar properties are expected to exist generally in large-scale wireless networks, depending on the network type and the degree of network penetration. For example, networks with lower level of coverage are probably less correlated with the population distribution. This is verified by our results for the UMTS network in Germany.

We presented also a novel analysis method for the impact of population densities on the spatial structure of wireless networks. More specifically, we estimated the number of required base stations for covering a given population pattern when the coverage ranges are changed, by employing the fractal dimension of the pattern. This estimation can be directly applied in several areas, e.g., in implementing cell splitting towards increasing the network capacity.

Based on our results we argue that population densities can be used as boot-strap information to generate more realistic network models for simulations and network analyses. This provides significantly better realism compared to simple assumption of uniform node distributions. Furthermore, it is possible to use this information in operational wireless networks. Topology information promises significant benefits for cognitive radios [16], [4]. However, having a knowledge about the network often requires additional system complexity and information exchange overhead. Therefore, we propose the utilization of information derived from population densities as a potential alternative for such applications. The spatial analysis methods which we have described yields quantitative and expressive results, and we emphasize that our approach needs not to be confined to a qualitative discussion, but several commercially viable and interesting applications can directly emerge from the statistical analysis.

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